Applications of Electromagnetic Phenomena in Electrical and Mechanical Systems
S.Yamada, A.Nafalski and S.C. Mukhopadhyay (eds.)


# Model-wavelets and their applications

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**ABSTRACT:** This paper proposes the modal-wavelet transform as one of the orthogonal wavelet transforms. The theoretical background and its application are described. The bases of modal-wavelet transform are derived from modal analysis of the potential field equations. Namely, the principal idea is that a numerical data set is regarded as a potential field. A modal matrix, constituting characteristic vectors, derived from discretized Poisson's equation enables us to carry out an orthogonal transformation like the discrete wavelets. The modal-wavelets based on the differential equation modeling yield efficient multi-resolution analysis. Applying its 3-dimensional analysis to a weather satellite infrared animation divides it into background and cloud moving images.

## 1 INTRODUCTION

The spread of high performance and reasonably priced computers has yielded a large scale Internet community as well as information resources. Data handling technologies based on the digital computers are of main importance to realize more efficient networking and computing. Discrete wavelet transform (WT) becomes a deterministic methodology to handle the digital signals and images, e.g., compressing data quantity, extracting their characteristics, etc [1]. Moreover, their applications to electromagnetic field calculation, solving for forward and inverse problems, have been investigated and spurred to faster calculation algorithm [2-3]. The conventional WT, however, sometimes suffers from limitation on subject data length which must be to the power of 2. Thereby, the applications depend on employed wavelet basis, and need an enormous memory installation for implementation. The principal purpose of this paper is to describe a new approach as the basis for more efficient wavelet analysis.

This paper proposes the modal-wavelet transform (MWT) as one of the WTs. The bases of MWT are derived from a modal analysis of the field of equations. Regarding a numerical data set as a potential field leads to a partial-differential-equation-based data modeling, i.e., the data set can be represented by Poisson's equations. Then, the modal analysis of the discretized Poisson's equation gives a modal matrix constituting characteristic vectors. The modal matrix enables us orthogonal transformation in the same way as WT. MWT uses this matrix as a wavelet basis. Because of the differential equation based modeling, MWT yields an optimal basis to the subject data length.

As an application, we demonstrate an animation analysis. The multi-resolution analysis of the animation frame axis classifies an animation into static and dynamic images. As a result, it reveals that MWT performs good computation efficiency in terms of memory usage.

## 2 MODAL-WAVELET TRANSFORM

### *2.1 Data representation by partial differential equations*

To derive a new wavelet basis we consider a data modeling approach based on the classical field theory. Namely, a numerical data set is assumed to be a potential field. According to the field theory, a scalar field $u$ caused by source density $\sigma$ can be represented by the Poisson equation:

$$\nabla^2 u = -\sigma . \qquad (1)$$

Discretization of Eq.(1) by numerical methods derives the following system of equations:

$$L\mathbf{U} = \mathbf{f} \qquad (2)$$

where **f** and **U** denote an input vector corresponding to the source density $\sigma$ and a solution vector representing the scalar field $u$ in Eq.(1), respectively; $L$ denotes a coefficient matrix corresponding to the Laplacian operator in Eq.(1). As an example, let each of pixel values in Fig.1(a) be a scalar potential, then the Laplacian operation to Fig.1(a) yields the source density distribution in Fig.1(b). Solving for Eq.(2) with the source density as vector **f** exactly reproduces the original image as shown in Fig.1(c). Therefore, our partial differential equation based modeling is capable of representing numerical data sets [4].

(a) Original image (128x128 pixels) (b) Source density (128x128 pixels) (c) Recovered image from Fig. 1(b)

Fig. 1. Image recovery from the source density by means of Poisson's equation

### *2.2 Modal-wavelet transform*

As is well known, the matrix $L$ in Eq.(2), derived by available discretizing methods, e.g., finite elements and finite differences, becomes a symmetrical as well as a positive definite matrix. In case when the vector **U** in Eq.(2) has $q$ elements, it is possible to obtain the characteristic values $\lambda_i$, $i$=1,2,...,$q$, of the matrix $L$ and their respective characteristic vectors $\mathbf{v}_i$, $i$=1,2,...,$q$. The matrix composed of the characteristic vectors $\mathbf{v}_i$, $i$=1,2,...,$q$ as its columns is called modal matrix:

$$M_q = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_q). \tag{3}$$

Because of the orthogonality, it holds the following relationship:

$$M_q{}^T M_q = I_q \tag{4}$$

where the superscript $T$ refers to a matrix transpose and $I_q$ is a $q$ x $q$ unit matrix. The modal matrix derived from the coefficient matrix of Laplacian operator has the same nature as those of the conventional WT matrices. Moreover, a linear combination of the characteristic vectors is possible to express the value distribution in a data set. Thus, MWT employs this modal matrix as WT matrices.

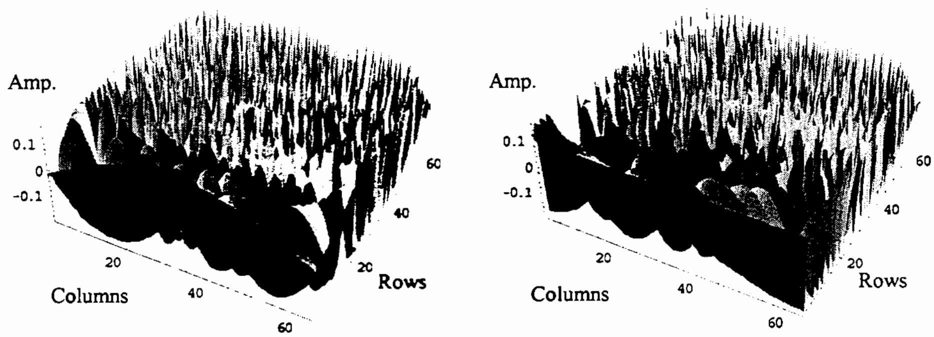
*2.3 Modal-wavelet transform matrix and basis*

The simplest MWT matrix is derived from the system matrix of the one-dimensional Laplacian operation with equi-meshed 3 points finite difference approximation. In the present paper, the matrix $L$ in Eq.(2) is realized by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} \cong U_{x-1} - 2U_x + U_{x+1}, \quad x = 1,2,\dots,q. \tag{5}$$

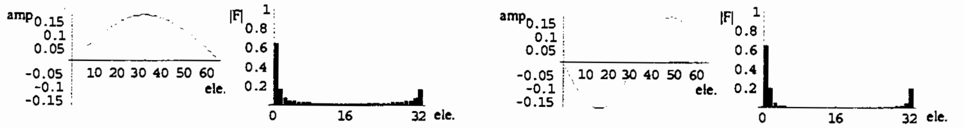
Then, applying the Jacobi method yields modal matrix $M_q$. Therefore, the dimension of matrix $M_q$ depends on the number of subdivisions of Eq.(5). This means it is possible to obtain an optimal basis to the subject data.

In the Laplace partial differential equations, two types of boundary conditions should be considered, i.e., the Dirichlet- and Neumann- type boundary conditions. Fig.2 illustrates the typical MWT matrices. As shown in Figs.3 and 4, the bases of the Dirichlet- and Neumann- type boundary conditions become odd- and even- functions, respectively. The bases of MWT look like sinusoidal functions, however, the bases are not composed of single frequency component. Moreover, the elements constituting the transform matrices never become the complex numbers like in the Fourier transform.




(a) Dirichlet type boundary condition (b) Neumann type boundary condition

Fig. 2. Modal-wavelet matrices (64 x 64)



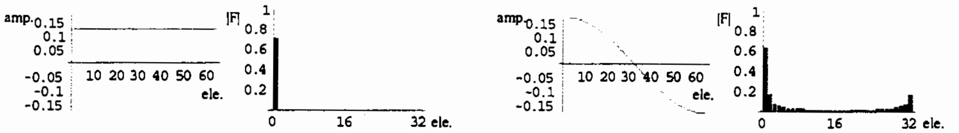

Elements Amplitude spectrum

(a) The first row vector

Elements Amplitude spectrum

(b) The second row vector

Fig. 3. Elements of row vectors in the matrix shown in Fig.2(a) and their Fourier amplitude spectrum




Elements Amplitude spectrum

(a) The first row vector

Elements Amplitude spectrum

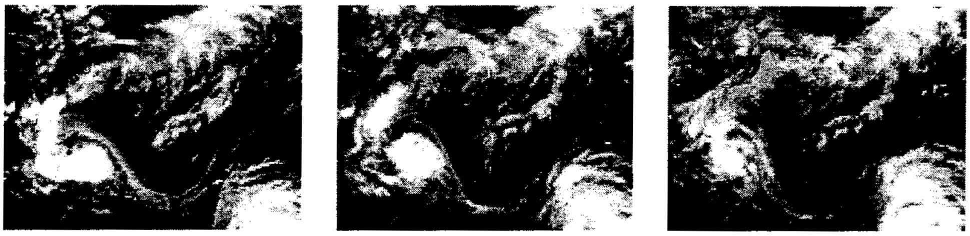
(b) The second row vector

Fig. 4. Elements of row vectors in the matrix shown in Fig.2(b) and their Fourier amplitude spectrum

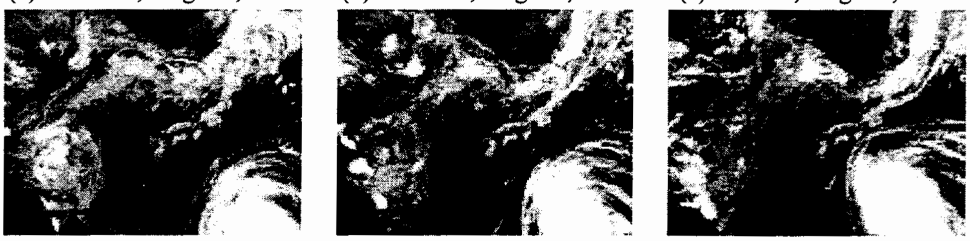
## 3 APPLICATION TO ANIMATION ANALYSIS

### *3.1 Infrared animation of weather satellite*

Fig.5 shows some frames of an infrared animation produced by weather satellite Himawari [5]. Applying MWT to this animation, separation of static and dynamic images is demonstrated. The animation used in this example is composed of 22 frames captured from 18:00 Aug. 10th to 15:00 Aug. 11th in 2000. Fig.5 indicates the generation process of typhoon No. 9 in 2000.



(a) At 18:00, Aug. 10, 2000 (b) At 22:00, Aug. 10, 2000 (c) At 2:00, Aug. 11, 2000



(d) At 6:00, Aug. 11, 2000 (e) At 10:00, Aug. 11, 2000 (f) At 14:00, Aug. 11, 2000

Fig. 5. Frames of infrared animation by weather satellite Himawari (256x193 pixels)

*3.2 3-dimensional modal-wavelet transform*

In order to apply MWT to the animation in Fig.5, the 3-dimensional MWT is essential to carry out. Namely, applying MWT to horizontal-, vertical- and frame- axes of the animation carries out animation analysis. Let us consider the animation $S_{lmn}$ having $m$ x $n$ pixels and $l$ frames. Then, its transpose rules are expressed by

$$[S_{lmn}]^T = S_{mnl},\ [S_{mnl}]^T = S_{nlm},\ [S_{nlm}]^T = S_{lmn}. \tag{6}$$

The 3-dimensional MWT gives the modal-wavelet spectrum $S_{lmn}'$:

$$S_{lmn}' = \left[M_n\left[M_m[M_l S_{lmn}]^T\right]^T\right]^T \tag{7}$$

where $M_l$, $M_m$ and $M_n$ are the $l$ by $l$-, $m$ by $m$- and $n$ by $n$- MWT matrices, respectively. And then, inverse MWT recovers the original animation $S_{lmn}$:

$$S_{lmn} = M_l^T\left[M_m^T\left[M_n^T\left[S_{lmn}'\right]^T\right]^T\right]^T. \tag{8}$$

Since a linear combination of weighted spectrum represents the original animation $S_{lmn}$, animation of each level can be generated by means of Eq.(8).

*3.3 Separation of static and dynamic images*

As shown in Figs.2(b) and 4(a), the lowest level of bases derived from Neumann type boundary condition is a constant term. This means that the multi-resolution analysis to the frame axis is capable of extracting the static term of animation when the Neumann type MWT matrix is employed. In much the same way, the dynamic term of animation can be extracted.

Figs.6 and 7 show the results of the multi-resolution analysis to the frame axis. Fig.6 is generated by means of Eq.(8) with only the lowest level of spectrum. In this case, the generated result has some frames, but all of frames are just the same as Fig.6. Thus, it suggests the extraction of background image and static air pressure distribution. On the other hand, Fig.7 shows dynamic term of animation generated by means of Eq.(8) without the lowest level of spectrum.

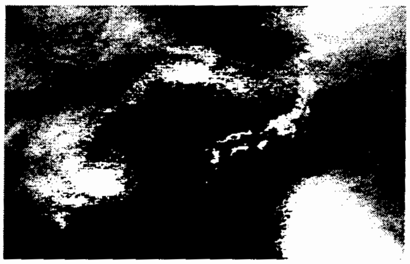
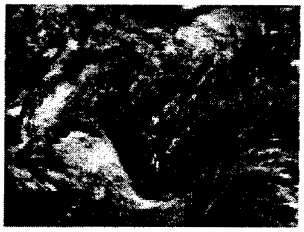
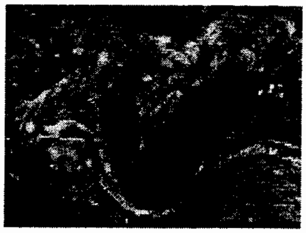
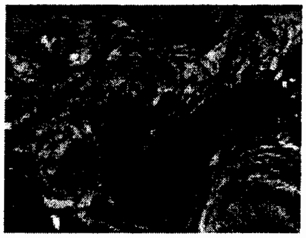
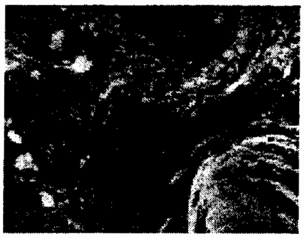
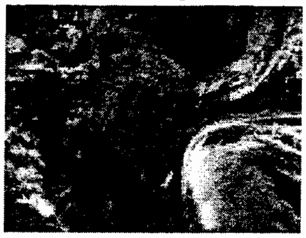


Fig. 6. Extracted static image (256 x 193 pixels)

(a) At 18:00, Aug. 10, 2000 (b) At 22:00, Aug. 10, 2000 (c) At 2:00, Aug. 11, 2000

(d) At 6:00, Aug. 11, 2000 (e) At 10:00, Aug. 11, 2000 (f) At 14:00, Aug. 11, 2000

Fig. 7. Frames of extracted dynamic image (256 x 193 pixels)

### *3.4 Comparison with conventional wavelets*

In the conventional WT, the data length $l$, $m$ and $n$ must be to the power of 2. In this animation analysis, the animation shown in Fig.5 has 256 x 193 pixels and 22 frames. If we carry out the same analysis with conventional WT, then $l$, $m$ and $n$ expressed in *Section 3.2* become 32, 256 and 256, respectively. In case of MWT, $l$, $m$ and $n$ are 22, 256 and 193, respectively. MWT accomplishes efficient analysis in terms of memory consumption.

## 4 CONCLUSIONS

We have proposed MWT and shown its application to animation analysis. Data representation by partial differential equations yields MWT bases optimal to subject data. Application to animation analysis has demonstrated separating static and dynamic images with high efficiency compared with conventional WT. As shown above, our MWT approach has versatile capability not only to information resource handling but also smart computing.

## 5 REFERENCES

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