IMAGE PROCESSING BY FIELD THEORY

–PART 1 : THEORITICAL BACKGROUND-

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Abstract. This paper proposes an innovative methodology of image processing based on the classical field theory. The key concept is that a pixel constituting a digital image is regarded as a kind of field potentials. Monochrome and color images are assumed to be scalar and vector potential fields, respectively. The vector operators for the image data, i.e., gradient, divergence and rotation, derive the image partial differential equations. Consequently, the static and dynamic images are represented by the Poisson and Helmholtz types of equations, respectively.

Keywords: Image processing, image vector calculus, image partial differential equations,

INTRODUCTION

Differentiation of image data is widely used in order to extract the edges of target objects in image processing. A computer screen composed of the pixels gives a physical meaning to the differentiation of the image data, i.e., spatial differentiation becomes a gradient operation when regarding the image data as the scalar potentials. This means that the differentiation of the image data yields the divergent vectors. In most physical systems, these vectors are called the field intensities. When we apply the divergent operation to the field intensities, it is possible to obtain the field source densities. In other words, solving for the Poisson type partial differential equation to these source densities, it is possible to obtain the potentials. Furthermore, taking into account the time varying field potential distribution reduces to the Helmholtz type partial differential equation. Its solution yields the potentials changing with time. As modern engineering as well as physics is mostly based on this potential field theory, so applying the field theory to image provides great advantages in the processing. This consideration motivates an application of the field theory to image processing.

All of images in digital computers are the discretized quantities in numerical values, and also the digital images are classified into monochrome and color images. The monochrome image is composed of two dimensional pixel array housing numerical values. On the other hand, the color image can be divided into RGB color components, i.e., *Red*, *Green* and *Blue*. Namely, the color image is composed of the pixels having three primal color components. This leads that monochrome and color images can be regarded as scalar and vector potential fields, respectively. This is a principal key idea of our image processing methodology proposed in this paper.

First section describes vector operations to monochrome image. This reveals that a static image can be reproduced as a solution of the Poisson type partial equation. Second, the color image is represented in terms of vector potential fields. As a result, a color image can be obtained the solutions of vector Poisson equations. Finally, animation generation by the Helmholtz type partial differential equation is discussed.

IMAGE PROCESSING BY FIELD THEORY

Monochrome Image and Vector Operations

The image processing by the field theory is started by regarding an image as potential field. Let a monochrome image be a scalar field U, then gradient operation to this scalar potential field U leads to a vector fields;

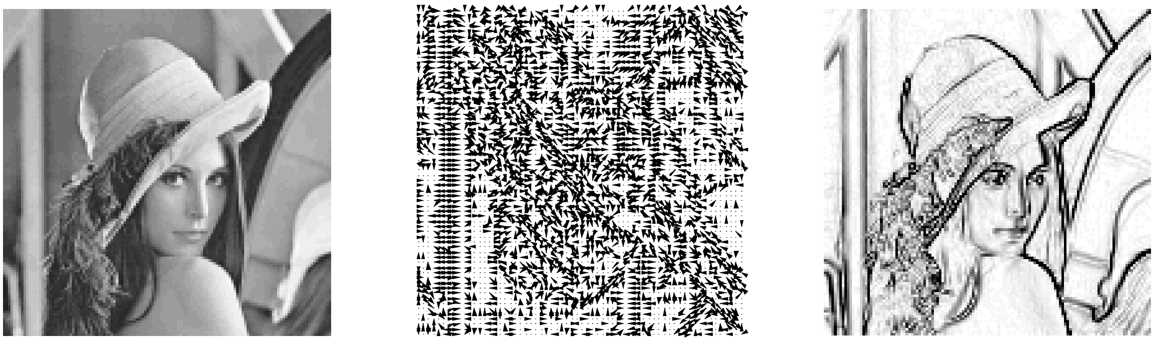
(1) $$\mathbf{V} = -\nabla U = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j},$$

where x and y respectively denote the horizontal- and vertical- direction axes on the image. Moreover, the vectors **i** and **j** are the unit directional vectors in x- and y-axes, respectively. Practical gradient operation in this image processing methodology is carried out by the central finite difference method. Namely, a pixel representing monochrome image is assumed to be a scalar potential $U_{i,j}$ obtained by finite difference discretization.

(2) $$\mathbf{V}_{i,j} = -\frac{U_{i+1,j} - U_{i-1,j}}{\Delta x}\mathbf{i} - \frac{U_{i,j+1} - U_{i,j-1}}{\Delta y}\mathbf{j},$$

where Δx and Δy denote the step-widths in x- and y- axes, respectively. Further the subscripts i and j in Eqn. 2 respectively refer to the positions i=1, 2,…, n in the x-axis and j=1, 2,…,m in the y-axis. The n and m are the resolutions in the direction of x- and y-axes, respectively. Fig. 1 shows (a) a monochrome image, (b) its vector distribution by gradient operation and (c) the sketch like image as an application of the gradient operation. The vectors distribute orthogonally to the edges along the target in Fig. 1(a). Calculating vector magnitude in each of vectors yields a sketch like image without any threshold techniques.

Divergent operation of the vector fields changes the vector quantities into the scalar quantities. Consequently, obtained scalar quantities are called the source densities, because they cause the vector fields. Divergent operation to the image vector **V** is given by,



(a) Original image having 128x128 pixels (b) Gradient operation to Fig. 1(a) (c) Sketch like image

Figure 1. Image gradient operation and its application.

$$\nabla \bullet \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) \bullet \left(V_x \mathbf{i} + V_y \mathbf{j} \right), \tag{3}$$

where V_x, V_y are the x- and y-components of the vector **V**. Thus, denoting the image source density by σ, the static monochrome image governing equation is derived as,

$$\nabla \bullet (-\nabla U) = -\nabla^2 U = -\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = \sigma . \tag{4}$$

Eqn. 4 is the Poisson equation. Practical Laplacian operator is replaced by the relevant finite differences. In this paper, we employ a nine-point finite difference formula:

$$\begin{aligned} \sigma_{i,j} = \frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{1}{6}[&U_{i-1,j-1} + U_{i-1,j+1} + 4U_{i-1,j} + 4U_{i+1,j} \\ &+ U_{i+1,j+1} + U_{i+1,j-1} + 4U_{i,j+1} + 4U_{i,j-1} - 20U_{i,j}], \end{aligned} \tag{5}$$

where the step-widths in the direction of x- and y-axes have been assumed to be 1. Also, zero Dirichlet boundary condition has been assumed at the edges of screen. Fig.2 (a) shows the source density of Fig. 1 (a). The Laplacian operation by Eqn. 5 removes the constant and first order spatial derivative terms from the image data. This means the Laplacian operation is capable of compressing the image data quantities, while the original image could be recovered from this source density σ. Modifying Eqn. 5 gives a system of equations,

$$\begin{aligned} U_{i,j} = \frac{1}{20}[&U_{i-1,j-1} + U_{i-1,j+1} + 4U_{i-1,j} + 4U_{i+1,j} \\ &+ U_{i+1,j+1} + U_{i+1,j-1} + 4U_{i,j+1} + 4U_{i,j-1} + 6\sigma_{i,j}]. \end{aligned} \tag{6}$$

Solving Eqn. 6 with the source density as input term is possible to obtain the original image. Fig. 2 (b) shows the recovered image from the source density shown in Fig. 2 (a). In case of employing the zero Dirichlet boundary condition at the edge of screen, the recovered image is identical to the original one. According to the nature of the finite differentials, employing fine mesh system enables us to generate a higher resolution image. Fig. 2(c) shows a high resolution image generated from source density shown in Fig. 2(a). As shown above, a static image can be represented by the Poisson equation. This means that application of the field theory to the digital images makes it possible to develop new image processing.

(a) Source density having 128x128 pixels

(b) Recovered from Fig. 2(a) to 128x128 pixels

(c) Generated from Fig. 2(a) to 256x256 pixels

Figure 2. Image recovery from the source density field by means of the Poisson equation.

Color image

Color image is composed of three primal color components, i.e., *Red*, *Green* and *Blue*. In such a case, a color image can be regarded as a vector potential field **A** having three orthogonal components, x, y and z. Curl curl equation of the vector potential field leads us to a differential equation as a governing equation,

(7) $$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} = \mathbf{J},$$

where color image **A** and its source density **J** are respectively defined by,

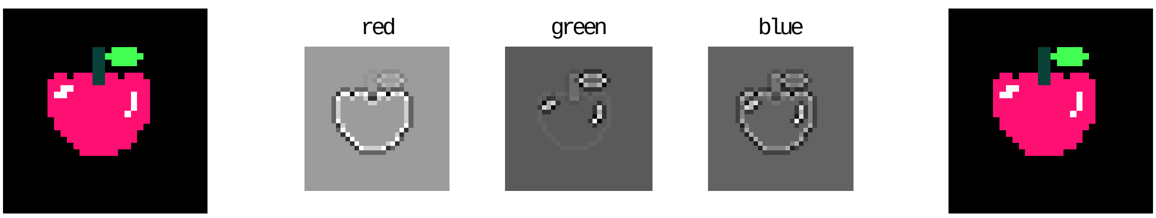
(8) $$\mathbf{A} = \mathbf{i}A_R(y,z) + \mathbf{j}A_G(z,x) + \mathbf{k}A_B(x,y), \quad \mathbf{J} = J_R\mathbf{i} + J_G\mathbf{j} + J_B\mathbf{k}.$$

The subscripts R, G and B respectively refer to *Red*, *Green* and *Blue* components. Moreover, **i**, **j** and **k** are the unit directional vectors in x-, y- and z-axes, respectively Let us assume a coulomb gauge to be satisfied as Eqn. 9, then three Poisson equations are derived as Eqn. 10 for each color component.

(9) $$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}A_R(y,z) + \frac{\partial}{\partial y}A_G(z,x) + \frac{\partial}{\partial y}A_B(x,y) \equiv 0.$$

(10) $$-\frac{\partial^2 A_R}{\partial y^2} - \frac{\partial^2 A_R}{\partial z^2} = J_R, -\frac{\partial^2 A_G}{\partial z^2} - \frac{\partial^2 A_G}{\partial x^2} = J_G, -\frac{\partial^2 A_B}{\partial x^2} - \frac{\partial^2 A_B}{\partial y^2} = J_Z.$$

This means color images can be independently evaluated in terms of RGB color source density components and parallel image processing is possible. In other words, each of color components can be handled in much the same speed as monochrome images.



(a) Original color image (8x8 pixels)

(b) Source densities J_R, J_G and J_B

(c) Recovered image (8x8 pixels)

Figure.3 Color image and its recovery

Animation generation by image Helmholtz equation

When we consider the time dependent images called animation, a Helmholtz type governing equation can be established. The Helmholtz type governing equation consists of the spatial as well as time derivative terms:

(11) $$\nabla^2 U + \frac{\partial}{\partial t}\alpha U + \frac{\partial^2}{\partial t^2}\beta U = -\sigma ,$$

where *t*, *α* and *β* are the time, the velocity and the repetitive moving speed parameters, respectively. The Helmholtz type equation is classified into two types. One is the diffusion equation when *β*=0, and the other is the wave equation when *α*=0. The former represents a spreading or shrinking animation, and the latter represents a vibrating or repetitive animation. Thereby, the image Helmholtz equation in Eqn. 11 is able to generate any types of animation.

After applying the state variable modification to the second time derivative terms in Eqn. 11, discretization of the equation yields a following system of equations:

(12) $$\left[S + \frac{\partial}{\partial t}H\right]\mathbf{X} = \mathbf{Y}\text{, or } \left[H^{-1}S + \frac{\partial}{\partial t}I\right]\mathbf{X} = H^{-1}\mathbf{Y},$$

where S and H are the $2 \times n \times m$ by $2 \times n \times m$ square matrices representing the Laplacian operator and coefficients *α* and/or *β*; **X** and **Y** are the vectors representing the animation frame and its source density, respectively. Also, I is a unit matrix. Even if *n* by *m* resolution, the state variable modification has led to the $2 \times n \times m^{\text{th}}$ system of equations. Let us consider the homogeneous equation of Eqn. 12,

(13) $$\left[H^{-1}S + \frac{\partial}{\partial t}I\right]\mathbf{X} = \mathbf{0}\text{, or } \left[\Gamma + \frac{\partial}{\partial t}I\right]\mathbf{X} = \mathbf{0},$$

then the normalized characteristic vectors $\mathbf{E}_i, i = 1,2,...,2 \times n \times m,$ of the matrix Γ are obtained. Consequently, diagonalization of Eqn. 12 can be carried out by the modal matrix Z constituted by the vectors $\mathbf{E}_i, i = 1,2,...,2 \times n \times m,$ as its column vectors, and then the general solution Eqn. 15 is obtained.

(14) $$\left(Z^{-1}\Gamma Z + \frac{\partial}{\partial t}\left[Z^{-1}IZ\right]\right)\mathbf{W} = Z^{-1}H^{-1}\mathbf{Y},$$

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 & \dots & w_{2xnxm} \end{bmatrix}^T,\ Z^{-1}H^{-1}\mathbf{Y} = \begin{bmatrix} g_1 & g_2 & \dots & g_{2xnxm} \end{bmatrix}^T$$

$$Z^{-1}\Gamma Z = \mathrm{diag}\left(\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_{2xnxm}\right), \quad Z^{-1}IZ = I,$$

(15) $$w_i = \left(w_{i0} - \frac{g_i}{\lambda_i}\right)\varepsilon^{-\lambda_i t} + \frac{g_i}{\lambda_i}, \qquad i = 1,2,...,2 \times n \times m,$$

where w_{i0} is an initial value of the coefficient w_i, which can be derived from the starting vector $\mathbf{X}_0$ representing stating image of animation, i.e.,

(16) $$\mathbf{X}_0 = Z'\mathbf{W}_0 = \sum_{i=1}^{2\times n\times m} w_{i0}\mathbf{E}_i,\ i = 1,2,...,2 \times n \times m.$$

The starting vector $\mathbf{X}_0$ is determined by a starting animation frame U_0. The final frame U_∞ is given in terms of the steady state solution vector $\mathbf{X}_\infty$. Thus, if the starting image U_0 is given in terms of the starting vector $\mathbf{X}_0$ and the final U_∞ images is given in terms of the input vector $\mathbf{Y}$, then any images between the times $t = 0$ and $t = \infty$ can be generated as the solution of Eqn. 11. Each of the local moving speeds depends on the selection of the parameters α and β. More precisely, the solution of Eqn. 12 is classified into three cases. The first is a simple time damping solution; the second is a damping or spreading oscillation; and the third is a pure oscillation solution. The characteristic values $\lambda_i, i = 1,2,...,2 \times n \times m$, of the first, second and third cases become the pure real, complex and pure imaginary numbers, respectively. Since each of the coefficients $w_i, i = 1,2,...,2 \times n \times m$, is possible to take the distinct characteristic value λ_i, then a locally moving animation can be generated. An example of animation generation is shown in Fig. 4. A lady wrinkling is obtained from two frames shown in Figs. 4 (a) and (e).

(a) Initial (b) Generated 1 (c) Generated 2 (d) Generated 2 (e) Final

Figure. 4 Animation generation by Helmholtz equation.

CONCLUSIONS

As shown above, we have proposed the new strategy for the image processing by means of the field theory. Regarding pixels representing an image as a kind of potential leads us to image vector operations. After that the image partial differential equations has been derived. The static and animation images can be obtained as the solution of the Poisson and the Helmholtz types of partial differential equations, respectively. This paper has described the solution strategies of them and we have succeeded in generating images with new methodology. Furthermore, handling of a monochrome and a color image has been described in our image processing strategy. As a result, it has been clarified that any images can be handled in much the same way as the field theory in the classical physics.

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