*Advanced Computational Electromagnetics*
*T. Honma (Ed.)*
*IOS Press, 1995*



# A STUDY OF INVERSE PROBLEMS IN ELECTROMAGNETIC FIELDS

**Tatsuya DOI and Yoshifuru SAITO**

**College of Engineering, Hosei University**
**Kajino Koganei, Tokyo 184, Japan**

This paper describes about the physical and mathematical backgrounds of the pilot point solutions on the sampled pattern matching method. Most of the inverse problem in electromagnetic fields reduces to solving for a following integral equation:

$$\mathbf{X} = \int_V G\mathbf{Y}dV,$$

where $\mathbf{X}, \mathbf{Y}, G$ and $V$ are the known field vector, unknown source vector, Green's function or its space derivative and volume containing the unknown source vector, respectively. Because of the difficulty to evaluate the exact solution vector $\mathbf{Y}$ from the locally measured field of $\mathbf{X}$, our sampled pattern matching method assumes that the magnitude of a source vector in each position can be represented by the space occupying rate of unit source vector. This means that the large and small source vectors are represented by the large and small number of unit source vectors in the problem region $V$, respectively. Physically, this transformation corresponds to the pulse width modulation technique in the power electronic engineering. Mathematically, this transformation is that the original governing equation is assumed to be modified

$$\mathbf{X}^{[N]} = \int_P G^{[N]}\delta dP,$$

where superscript [N] refers to the normalized quantities. Also, $\delta$ and $P$ are the vector delta function representing the source vector $\mathbf{Y}$ and $(|G|/|\mathbf{X}|)V$, respectively. A methodology to decide the existence of the vector delta function is one of the key ideas of the sampled pattern matching method.

## 1. INTRODUCTION

With the developments of modern digital computers, numerical methods, e.g., finite element, finite difference and boundary element methods, for the regular problems have been developed and spread as one of the most effective analyzing tools for the engineering as well as physical science. As a result, we are currently available various commercially based packages for the regular problem analysis. When we employ the existing numerical approaches for the inverse problems, a large number of try and error iterations should be carried out. This spurs the development of the methodology for the inverse problems.

Further, development of the high sensitive magnetic field measurement device based on the superconducting quantum effect, i.e., SQUID flux meter, stimulates the establishment of the inverse analysis methodology, because the human heart and brain diagnosis from the local magnetic field measurement essentially require the solution of the inverse problems.

Previously, we have proposed the sampled pattern matching (SPM) method for the inverse problems in the electromagnetic fields. Successful results were obtained by our SPM method for the medical as well as non-destructive testing applications. Particularly, we have succeeded in searching for the Wolf-Parkinson-White (WPW) syndromes from the magnetocardiograms, and the plural defects in the conductive materials [1-8].

This paper describes about the physical and mathematical backgrounds of the pilot point solutions on the SPM solution procedure. A key idea of the SPM method is that the magnitude of a source vector in each position is assumed to be represented by the space occupying rate of unit source vectors. This means that the large and small source vectors are transformed into the large and small number of unit source vectors in the problem region, respectively. Physically, this transformation corresponds to the pulse width modulation (PWM) technique in the power electronic engineering. Mathematically, this transformation is that the solution of the inverse problems is assumed to be represented by a combination of the vector delta functions. A simple example shows the concrete process of the SPM method.

## 2. INVERSE PROBLEMS IN ELECTROMAGNETIC FIELDS

### 2.1. Classification of the inverse problems

Before to move on a general case, let us consider a simple magnetostatic field as a starting point. Assuming the Coulomb gauge, a governing equation for the magnetostatic field system is

$$\left(\frac{1}{\mu}\right)\nabla^2\mathbf{A} = -\mathbf{J}_S, \tag{1}$$

where $\mathbf{A}$, $\mathbf{J}_S$ and $\mu$ are the magnetic vector potential, source current density and magnetic permeability, respectively. This equation can be modified by introducing the magnetic permeability of air $\mu_0$ as

$$\begin{aligned}\left(\frac{1}{\mu_0}\right)\nabla^2\mathbf{A} &= -\mathbf{J}_S - \left[\left(\frac{1}{\mu}\right) - \left(\frac{1}{\mu_0}\right)\right]\nabla^2\mathbf{A} \\ &= -\mathbf{J}_S - \mathbf{J}_E,\end{aligned} \tag{2}$$

where $\mathbf{J}_E$ is an equivalent current density caused by medium discontinuity. Imposing a homogeneous open boundary condition to (2), we have

$$\mathbf{A} = \int_V G\mu_0\mathbf{J}_S dV + \int_V G\mu_0\mathbf{J}_E dV, \tag{3}$$

where $G$ is a Green's function. It must be noted that the equivalent current density $\mathbf{J}_E$ is a function of the vector potential $\mathbf{A}$, also this vector potential $\mathbf{A}$ is a function of the source current density $\mathbf{J}_S$. Thereby, we have

$$\mathbf{J}_E = f(\mathbf{J}_S). \tag{4}$$

Likewise any electromagnetic fields can be reduced to solving for a following type integral equation:

$$\mathbf{X} = \int_V G\mathbf{Y}_S dV + \int_V G\mathbf{Y}_E dV, \tag{5}$$

where $\mathbf{X}, \mathbf{Y}_S, \mathbf{Y}_E$ and $G$ are the field vector, field source, equivalent field source caused by medium discontinuity and Green's function or its space derivative, respectively. Similar reason to those of (4), following relationship is held:

$$\mathbf{Y}_E = f(\mathbf{Y}_S). \tag{6}$$

***a)Inverse parameter problems*** At first, let us consider a problem that a part vector $\Delta\mathbf{X}_P$ of the entire field vector $\mathbf{X}$ and field source $\mathbf{Y}_S$ are given, then we have to evaluate a medium parameter in the problem region. From (5), it is obvious that the field vector $\mathbf{X}$ is caused by the two input field sources $\mathbf{Y}_S$ and $\mathbf{Y}_E$, i.e.,

$$\mathbf{X}_S + \mathbf{X}_E = \int_V G\mathbf{Y}_S dV + \int_V G\mathbf{Y}_E dV, \tag{7}$$

where the field vectors $\mathbf{X}_S$ and $\mathbf{X}_E$ correspond to the first and second terms on the right, respectively. From (5) and (7), it is possible to obtain a following relationship:

$$\begin{aligned} \Delta\mathbf{X}_E &= \Delta\mathbf{X}_P - \Delta\mathbf{X}_S \\ &= \int_V G\mathbf{Y}_E dV, \end{aligned} \tag{8}$$

where $\Delta\mathbf{X}_E$ is a part vector of $\mathbf{X}_E$ in (7) and $\Delta\mathbf{X}_S$ is a part vector of $\mathbf{X}_S$ for which is calculated from the given field source $\mathbf{Y}_S$ imposing no medium discontinuous condition. The problem governed by (8) is called the *inverse parameter (IP) problems*. Because of (6), if a set of reasonable part vectors $\Delta\mathbf{X}_p$ is given by scanning the field source $\mathbf{Y}_S$, then this IP problem can be solved uniquely. Most of the non-destructive testing for defect identification in the metallic materials is reduced into this type IP problem [5-7].

***b) Inverse source problems*** Secondly, let us consider a problem that a part vector $\Delta\mathbf{X}_p$of the entire field vector $\mathbf{X}$ in (5) is given and there is no medium discontinuity, then we have to evaluate the field source $\mathbf{Y}_S$. No medium discontinuity means no equivalent field source $\mathbf{Y}_E$ in (5). This leads to a following governing equation:

$$\Delta\mathbf{X}_P = \int_V G\mathbf{Y}_S dV. \tag{9}$$

The problem governed by (9) is called the *inverse source (IS) problems*. Because of the lack of information, it is difficult to evaluate a unique solution $\mathbf{Y}_S$ for this problem. But most of the valuable inverse problems to be solved become this type IS problem. Also, the IP problem without reasonable information is fallen into this type problem. Thus, we have to attack the inverse source problem.

### 2.2. Sampled pattern matching method

A key idea of the SPM method is that the magnitude of a source vector in each position is assumed to be represented by the space occupying rate of unit source vectors. This means that the large and small source vectors are transformed into the large and small number of unit source vectors in the problem region, respectively. Physically, this transformation corresponds to the PWM technique in the power electronic engineering. Mathematically, this transformation is that the solution of the inverse problems is assumed to be represented by a combination of the unit vector delta functions.

***a) Physical concept of the SPM method*** As shown in Fig. 1, the fields are spreading from the field source distributing space and only the limited local fields can be measured. Since we are given only a part vector $\Delta\mathbf{X}_P$ of the entire field vector $\mathbf{X}$ in (9), it is difficult to obtain an exact field source $\mathbf{Y}_S$. Thus, we try to represent the source vector pattern in terms of a combination of unit pulse. Figures 2(a) and 2(b) show an example of original field source pattern and its PWM representation, respectively.

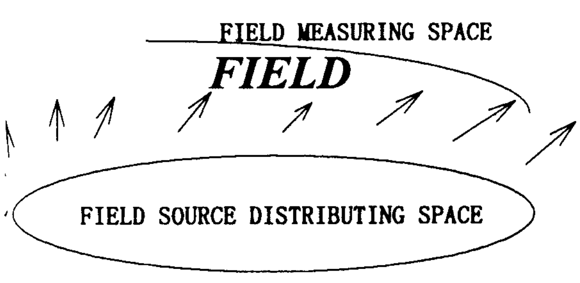



Figure 1. Schematic diagram of an inverse source problem.

Consideration of Fig.2 suggests that the PWM representation makes it possible to transform the amplitude of original field source into the concentrating ratio of the unit pulse in position. This means that it is impossible to evaluate the amplitude of field source but possible to evaluate the position of unit pulse from the known field vector $\Delta\mathbf{X}_P$.

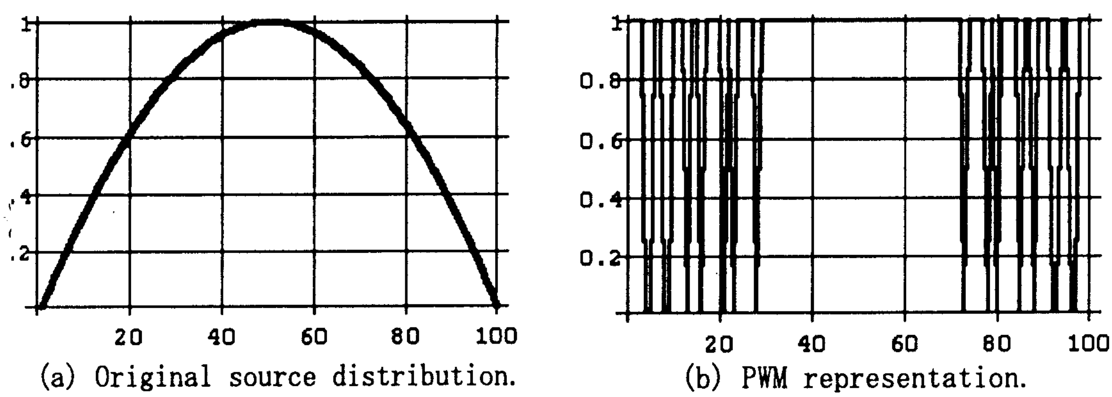
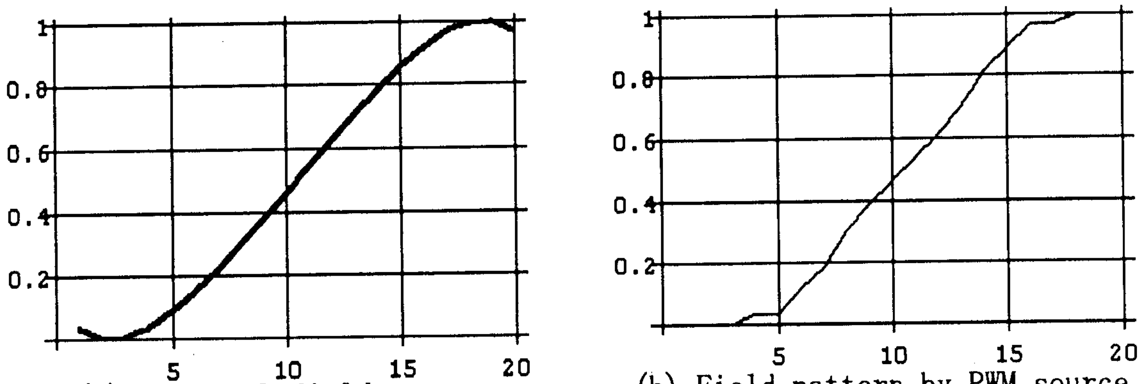
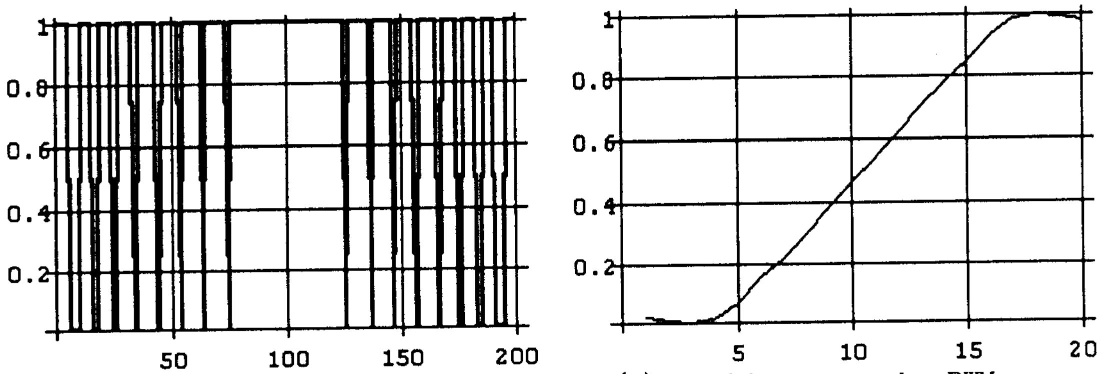



Figure 2. An example of the PWM representation of the field source $\mathbf{Y}_S$ pattern.(a) Original field source pattern $\mathbf{Y}_S$, and (b) its PWM representation.




(a) Original field pattern. (b) Field pattern by PWM source.

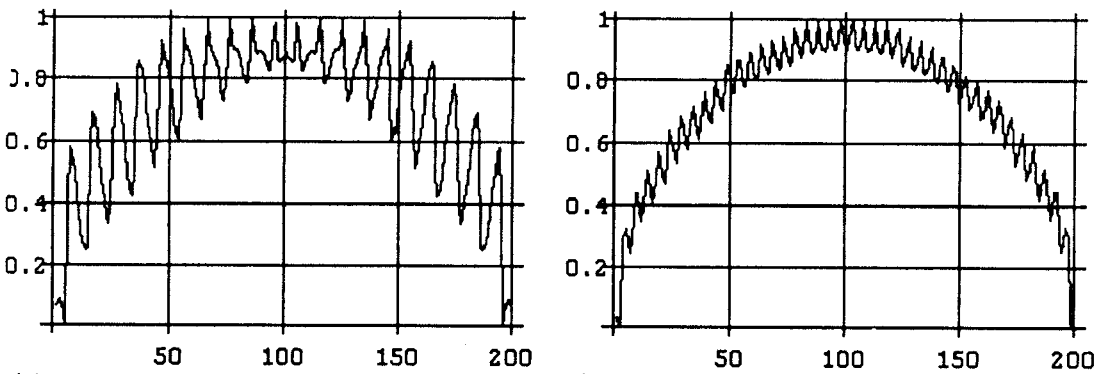
Figure 3. Originally given field pattern of $\Delta \mathbf{X}_P$ and those by PWM field source. (a) Originally given field pattern of $\Delta \mathbf{X}_P$, and (b) the field pattern given by the PWM field source.




(a) PWM field source representatic (b) Field pattern by PWM source.

Figure 4. Improvement by increasing the number of pulses. (a) PWM field source representation, and (b) field pattern caused by the PWM field source employing twice number of pulses to the Figs. 2(b) and 3(b).

Probably every one will has a question about a difference between the originally given field $\Delta \mathbf{X}_P$ and those given by the PWM field source. Figures 3(a) and 3(b) show the originally given field vector pattern of $\Delta \mathbf{X}_P$ and those given by the PWM field source, respectively. As a result of Fig. 3, it is revealed that the field can be reproduced not only by the original field source $\mathbf{Y}_S$ in Fig. 2(a) but also by the PWM field source in Fig. 2(b). Also, every one would like to know the result when increasing the number of pulses in Fig. 2(b). Figure 4(a) and 4(b) show the PWM field source and its field pattern employing twice number of pulses (200) to those of Figs. 2(b). Obviously, the field pattern in Fig. 4(b) is fairly improved comparing with those of




(a) Inverted field source pattern from 20 field points.

(b) Inverted field source pattern from 40 field points.

Figure 5. Inverted field source patterns. (a) Field source pattern inverted from 20 field measured points, and (b) field source pattern inverted from 40 field measured points.

Fig. 3(b).

Finally, is it possible to invert from the PWM field source pattern to the original field source pattern ? Answer of this question is yes but only approximately possible. This is because the governing equation (9) never yield a unique solution. Figure 5 shows the inverted field source patterns. Left- and right-side results were inverted from 20 and 40 field measured points, respectively. Comparison the results in Fig. 5 with the exact field source pattern in Fig. 2(a) suggests that the number of field measured points [i.e., the order of field vector $\Delta \mathbf{X}_P$ in (9)] reflects to the accuracy of the inverted results.

***b) Mathematical background*** As shown above, our SPM method transforms the original field distribution $\mathbf{Y}_s$ in Fig. 2(a) into the PWM field source pattern $\mathbf{P}$ in Fig. 2(b) using the given field $\Delta \mathbf{X}_P$. This transformation is carried out in a following way. At first, the governing equation (9) is assumed to be modified into

$$\Delta \mathbf{X}_P^{[N]} = \int_P G^{[N]} \delta dP, \tag{10}$$

where superscript [N] refers to the normalized quantities. Also, $\delta$ and $P$ are the vector delta function representing the source vector $\mathbf{Y}_s$ and $(|G| / |\Delta \mathbf{X}_P|)V$, respectively.

Discretizing (10), we have

$$\Delta \mathbf{X}_{\mathrm{P}}^{[N]} = \sum_{i=1}^{m} \Delta P_i \mathbf{G}_i^{[N]} = D\mathbf{P}, \tag{11}$$

where $m$ denotes a number of subdivisions of the field source existing space. Denoting $n$ as a number of field measured points (i.e., the order of vector $\Delta\mathbf{X}_{\mathrm{P}}$), $D$ becomes a $n$ by $m$ rectangular system matrix composed by the column vectors $\mathbf{G}_i^{[N]}(i = 1,2,\cdot,m)$, and $\mathbf{P}$ is a $m$-th order PWM field source vector whose element is $\Delta P_i (i = 1,2,\cdot,m)$. In order to evaluate the vector $\mathbf{P}$ in a least squares sense, multiplication of $D^T$ to the both sides of (11) yields

$$D^T \Delta\mathbf{X}_{\mathrm{P}}^{[N]} = D^T D\mathbf{P}, \tag{12}$$

or

$$\mathbf{P} = \left|D^T D\right|^{-1} D^T \Delta\mathbf{X}_{\mathrm{P}}^{[N]}. \tag{13}$$

From (13), it seems to be able to evaluate the solution vector $\mathbf{P}$. But, this is practically difficult, because the column vectors $\mathbf{G}_i^{[N]}(i = 1,2,\cdot,m)$ constituting the system matrix $D$ are not linear independent. In the other words, the elements of matrix $D$ have been obtained by discretizing the same continuous function $G$, so that the matrix $D^T D$ becomes a singular matrix. Thus, it is difficult to evaluate the vector $\mathbf{P}$ by means of the conventional least squares fit.

Consideration of the matrix $D^T D$ in (12) reveals that the diagonal elements take 1 but the other elements are always less than 1 because each of the column vectors $\mathbf{G}_i^{[N]}(i = 1,2,\cdot,m)$ in $D$ is normalized. Thereby, the matrix $D^T D$ may be regarded a unit matrix with order $m$. This assumption means that (12) yields an approximate solution of $\mathbf{P}$, which coincides with those of a factor analysis. Further consideration of (12) suggests that the elements in (12) take the values between -1 and 1. Namely, the elements $\Delta P_i, (i = 1,2,\cdot,m)$ in the vector $\mathbf{P}$ are

$$\Delta P_i = \frac{\Delta\mathbf{X}_{\mathrm{P}}^T \cdot \mathbf{G}_i}{|\Delta\mathbf{X}_{\mathrm{P}}||\mathbf{G}_i|}, \qquad i = 1,2,\bullet,m, \tag{14}$$

where the elements $\Delta P_i, (i = 1,2,\cdot,m)$ of $\mathbf{P}$ are called the *pattern matching figures*.

Since we have to decide the existence of vector delta function $\delta$ in (10) by the least squares sense, the SPM method assumes that only one element taking the maximum pattern matching figure in (12) have a unit vector delta function. If the $\Delta P_h$ takes the maximum, then this point $h$ is the *first pilot point* and its associated pattern vector $\mathbf{G}_h$ is called the *first pilot pattern*. To decide the second pilot point, let us assume that (11) is modified to

$$\Delta \mathbf{X}_P^{[N]} - \sum_{i=1}^{m} \Delta P_i [\mathbf{G}_h + \mathbf{G}_i]^{[N]} \\ = D'\mathbf{P}', \tag{15}$$

where $D'$ and $\mathbf{P}'$ are the $n$ by $m$-1 rectangular matrix and vector with order $m$-1. Similar to the first pilot point searching process, the second pilot point is decided as the maximum element of

$$[D']^T \mathbf{X}_P^{[N]} - [D']^T D'\mathbf{P}'. \tag{16}$$

Continuing the similar processes of (11)-(16) until the peak value of pattern matching figure is obtained, the field source $\mathbf{Y}_S$ is transformed into the PWM field source pattern $\mathbf{P}$. This transformation is called the *SPM transformation*. Also, obtained solution is called the *pilot point solution* or *PWM solution*.

It must be noted here that the SPM transformation can be carried out in the two different ways. One is the adding process of a unit vector delta function starting the zero field source condition $[\Delta P_i = 0, (i = 1, 2, \cdot, m)]$, and the other is the deleting process of a unit vector delta function starting the full field source condition $[\Delta P_i = 1, (i = 1, 2, \cdot, m)]$. Practical examples of the PWM solutions shown in Figs. 2(b) and 4(a) were evaluated by the deleting process, because the deleting process yielded a larger peak pattern matching figure. Figures 6(a) and 6(b) show the convergence of the pattern matching figures for the adding and deleting processes, respectively. The maximum pattern matching figures for the adding and deleting processes were 0.982 and 0.999, respectively.

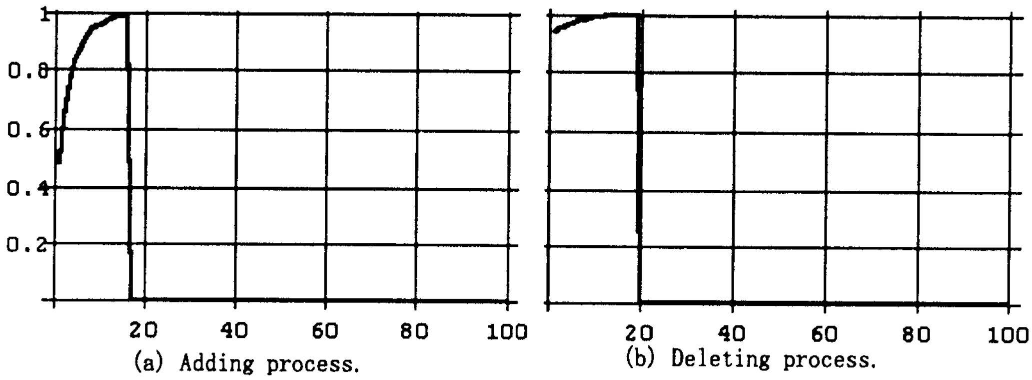
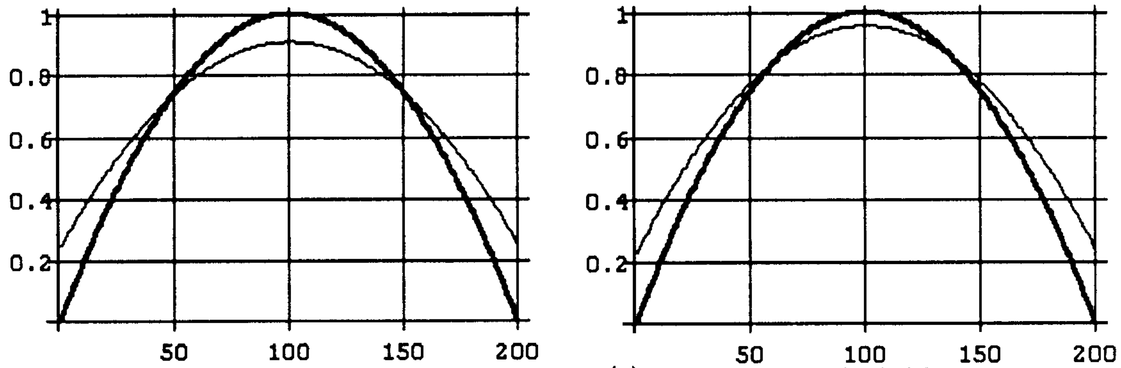



Figure 6. Convergence of the pattern matching figure. Vertical and horizontal axes are the pattern matching figure and number of pilot points, respectively. (a) Adding process and (b) deleting process.




(a) Post-processed field source pattern from 20 field points.

(b) Post-processed field source pattern from 40 field points.

Figure 7. Post-processed spectrum solutions by the second order least squares fit. Solid and thin lines refer to the exact and evaluated field source patterns, respectively. (a) Field source pattern inverted from 20 field measured points, and (b) field source pattern inverted from 40 field measured points.

The field pattern $\Delta\mathbf{X}_P{}'$ caused by the PWM solution **P** is simply evaluated by substituting the PWM solution **P** into Eq. (11). Practically evaluated examples of the field patterns $\Delta\mathbf{X}_P{}'$ were shown in Figs. 3(b) and 4(b).

The inversion from the PWM solution **P** to the original field source pattern **S** is carried out by

$$\mathbf{S} = D^T[\mathbf{X}_P{}']^{[N]}. \tag{17}$$

The solution **S** given by (17) is called the *spectrum solution* whose examples have been shown in Figs. 5(a) and 5(b). Further, it is obvious that the spectrum solutions **S** in Figs. 5(a) and 5(b) can be fitted by the second order function of position. Figures 7(a) and 7(b) show the second order least squares fit of the spectrum solutions in Figs. 5(a) and 5(b), respectively. From Fig. 7, it is observed that increasing the number of field measured points improves the solution.

## 3. CONCLUSION

As shown above, this paper has clarified the physical as well as mathematical backgrounds of our SPM method. As a result, the SPM method is a nonlinear transformation method based on the least squares fit. Thus, the SPM method makes it possible to evaluate an approximate solution of the inverse source problems.

The SPM solution process is summarized as follows.

1. Evaluate the PWM solution pattern **P** based on the approximate least squares sense.
2. Evaluate the field pattern $\Delta \mathbf{X}_P'$ from the PWM solution **P**.
3. Evaluate the spectrum solution **S** by the approximate least squares sense.
4. Post processing the spectrum solution.

Implementation of above SPM process is easily carried out by the Mathematica. All of the examples shown in this paper were implemented by the Mathematica.

## REFERENCES

[1]Y. Saito, et al., J.Appl. Phys., Vol. 67, No.9, May 1990, pp.5830-5832.

[2]Y. Saito, et al., Nonlinear Phenomena in Electromagnetic Fields, T.Furuhashi and Y.Uchikawa (Editors), Elsevier Pub., 1992, pp. 185-188.

[3]H.Saotome, et al., Int. J. Appl. Electmagn. Maters., Vol. 3, 1993, pp. 297-306.

[4]H. Saotome, et al., IEEE Trans. Magn. Vol. MAG-29, No. 2, 1993, pp. 1389-1394.

[5]H. Saotome, et al., IEEE Trans. Magn. Vol. MAG-29, No. 2, 1993, pp. 1861-1864.

[6]T. Doi, et al., J.Appl. Phys., in printing.

[7]T. Doi, et al., IEEJ-A, Vol. 113, No.4, 1994, pp. 290-295.

[8]K. Kitsuta, et al., Nonlinear Phenomena in Electromagnetic Fields, T.Furuhashi and Y.Uchikawa (Editors), Elsevier Pub., 1992, pp. 77-80.