Faster open-boundary magnetic field computation using the strategic dual image and Voronoi–Delaunay transformation methods

Y. Saito, Y. Nakazawa, and S. Hayano
College of Engineering, Hosei University, 3-7-2 Kajinocho, Koganei, Tokyo 184, Japan

In order to obtain the finite element solutions of open-boundary magnetic field problems, we have previously proposed a strategic dual image (SDI) method. This SDI method requires the use of a specific shape of hypothetical boundary so that it sometimes requires a solution of a king size system of equations. To remove this deficiency, this paper proposes an efficient implementation technique of the SDI method based on a Voronoi–Delaunay transformation. As a result, it is revealed that a king-size system of equations can be dramatically reduced to a small size system of equations.

I. INTRODUCTION

The finite-element method is being widely used to solve various engineering and physical field problems. Most of the engineering and physical field problems have a more or less open boundary so that the finite-element method confronts a serious difficulty. To overcome this difficulty, various means have been proposed. The methods are roughly classified into two categories: one is based on the combination of finite- and boundary-element methods; the other is the infinite and exterior finite-element methods.[1-5] In spite of these efforts, it is still required to invent a deterministic method for the open-boundary problems because the existing methods require considerable computer time and programming effort compared with those of the traditional finite-element method.

Previously, we proposed a strategic dual image (SDI) method to evaluate the finite element solutions of open-boundary problems in an extremely simple manner.[6-8] This SDI method is not a numerical technique but an analytical one for open-boundary problems, so that even finite difference solutions of open-boundary problems could be evaluated by means of the SDI. Even though the SDI method is an extremely simple and effective procedure for the open-boundary problems, it sometimes becomes an expensive task because the SDI method inevitably requires the use of a specific shape of hypothetical boundary.

In this paper we propose an efficient implementation technique of the SDI method based on the Voronoi–Delaunay (VD) transformation. In the other words, the VD transformation method which has been exploited to solve the closed-boundary problems in an ultimate efficient manner is now applied to open-boundary problems.[9-11] As a result, it is revealed that a king-size system of equations can be dramatically reduced to a small-size system of equations.

II. THE SDI SOLUTIONS USING THE VD TRANSFORMATION METHOD

A. Strategic dual image method

The key idea of SDI method is that any vector fields can be divided into two components: rotational and divergent; and their field sources are also divided into two types: rotational and divergent. Thereby, it is possible to exploit a method by which the rotational and divergent components can be obtained by imposing the rotational and divergent field source images, respectively. In magnetic field problems the rotational and divergent field sources are, respectively, corresponding to the current i and magnetic charge m so that the rotational field component can be obtained by imposing the corresponding image current, as shown in Fig. 1(a). In this case, the following condition:

$$\sum_{p=1}^{q} \frac{i_p}{r_p} = 0, \tag{1}$$

must be satisfied at the center of a circular hypothetic boundary to reduce the net image to zero. In Eq. (1), q and r_p denote the number of source currents and the distance from a center of circular hypothetic boundary to the current i_p, respectively. Equation (1) and the image in Fig. 1(a) mean that the total currents in the problem region must be zero, and the vector potentials at the center as well as on the circular hypothetic boundary must be zero. Therefore, the

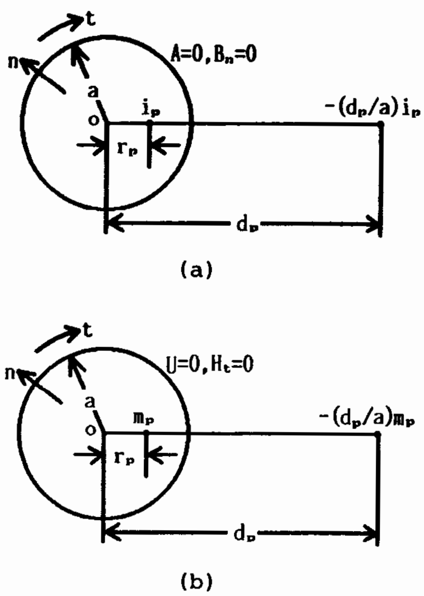

FIG. 1. (a) The rotational field source image $-(d_p/a)i_p$ and hypothetic boundary. (b) The divergent field source image $-(d_p/a)m_p$ and hypothetic boundary.

calculation of the rotational field component can be reduced to the solution of a vector potential problem having zero boundary conditions at the circular hypothetic boundary and the center of its hypothetic boundary. Similarly, it is possible to show that the calculation of the divergent field component can be reduced to the solution of a scalar potential problem having zero boundary conditions at the circular hypothetic boundary and the center of hypothetic boundary, as shown in Fig. 1(b). Obviously, the zero boundary condition of scalar potential U at the hypothetic boundary corresponds to the symmetrical boundary condition of vector potential A. This means, if we employ the vector potential A to represent the open field, then the calculation of the divergent field component may be reduced to the solution of a vector potential problem having the symmetrical boundary condition at the circular hypothetic boundary and the zero boundary condition at the center of hypothetic boundary. Thus, the open-boundary field calculation can be reduced to the solution of zero and symmetrical boundary problems having the circular hypothetic boundary in two dimensions. Furthermore, the zero condition must be set at the center of circular hypothetic boundary by (1) for both zero and symmetrical boundary problems.

B. Voronoi–Delaunay transformation method

The VD transformation is based on a VD discretization method. The key idea of the VD discretization is that a dual-energy finite-element method is implemented by means of a geometrical duality between the Voronoi polygons and associated Delaunay triangles using a single potential.[12,13] The conventional dual-energy finite-element method requires the use of two different types of potentials so that the functionals can be obtained in a most efficient manner but it is difficult to obtain the improved local solutions.[14] This deficiency was removed by the VD discretization method. In the two-dimensional magnetostatic fields of Fig. 2, the nodal equation of the Delaunay system between nodes i and j is given by[12]

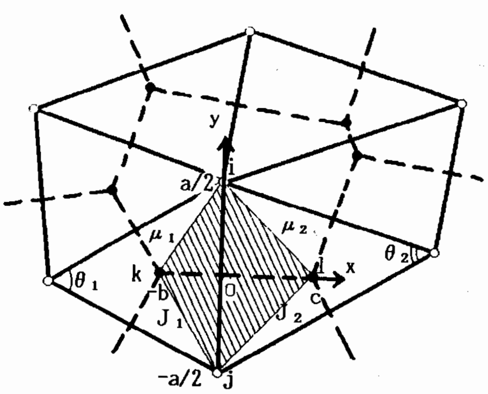

FIG. 2. Voronoi–Delaunay diagram and locally orthogonal coordinate system. Solid line denotes the Delaunay triangles; dotted line denotes the Voronoi polygons; and the vertices of Voronoi polygons are located at the circumcenters of Delaunay triangles.

$$\left(\frac{\cot\theta_1}{2\mu_1} + \frac{\cot\theta_2}{2\mu_2}\right)(A_i - A_j) = \frac{a}{4}(bJ_1 + cJ_2), \quad (2)$$

where A_i and A_j are the nodal potentials located at nodes i and j, respectively. Also, angles θ_1, θ_2, permeabilities μ_1, μ_2, current densities J_1, J_2, and lengths a,b,c are shown in Fig. 2. On the other side, the nodal equation of the Voronoi system between the nodes k and l in Fig. 2 is given by[12]

$$(A_k - A_l)\Big/\left(\frac{\mu_1}{2}\cot\theta_1 + \frac{\mu_2}{2}\cot\theta_2\right) = \frac{ab}{2}J_1, \quad (3)$$

where A_k and A_l are the nodal potentials located at nodes k and l, respectively. If the permeabilities in (2) and (3) are set to a normalized value, viz., $\mu_1 = \mu_2 = 1$, then a product of the geometrical coefficients in (2) and (3) gives

$$\left(\frac{\cot\theta_1}{2} + \frac{\cot\theta_2}{2}\right)\Big/\left(\frac{\cot\theta_1}{2} - \frac{\cot\theta_2}{2}\right) = 1. \quad (4)$$

Equation (4) reveals that a geometrical duality between the Delaunay and Voronoi systems can be established in the Voronoi–Delaunay discretization because the geometrical coefficient of potentials A_i, A_j in the Delaunay system is a reciprocal to those of potentials A_k, A_l in the Voronoi system. In the network terminologies, the Delaunay system is a reciprocal network of the Voronoi system.[15] This remarkable relationship leads to the VD transformation method for which the Delaunay solution vector Φ_D can be represented in terms of the Voronoi solution vector Φ_V by

$$\Phi_D = C\Phi_V, \quad (5)$$

where C is a transformation matrix. This transformation matrix C can be derived by satisfying a Laplace equation at the vertices of the Delaunay triangles. Details of this derivation are described in Refs. 9–11. Thus, the functional of the dual-energy approach can be evaluated by averaging the Delaunay and Voronoi system functionals using only the Voronoi system solution vector Φ_V. Also, by means of Eq. (5), improved local solutions can be obtained by interpolating the potentials located at the midpoints between the vertices of Delaunay triangles and Voronoi polygons.[9–13]

C. Open-boundary solutions

When the symmetrical boundary solution vector X_s and the zero boundary solution vector X_z are obtained after solving each of their systems, the open-boundary solution vector X is formally obtained by

$$X = (1/2)[X_s + X_z], \quad (6)$$

where a coefficient (1/2) comes from the two field sources, viz., the rotational and divergent field sources. At the hypothetical boundary, (6) is reduced to $X = (1/2)X_s$ because the other solution vector X_z is always zero. By means of this and (6), a practical open-boundary solution can be obtained from a following system equation:

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & 2D_{22} - D_{21}D_{11}^{-1}D_{12} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} F_s \\ 0 \end{vmatrix}, \quad (7)$$

where X_1 is a subvector on the inside region; X_2 is a subvector on the hypothetical boundary; F_s is an input source vector; and submatrices $D_{11}, D_{12}, D_{21}, D_{22}$ are correspondingly

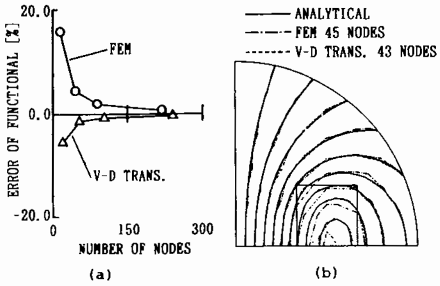

FIG. 3. (a) An example of convergence property of the functionals. (b) An example of the local field distributions.

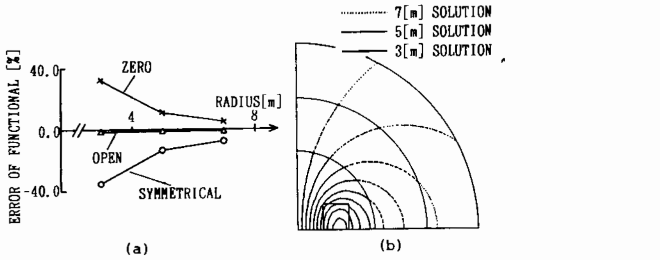

FIG. 4. (a) The convergence property of the functionals when the different radii of the hypothetical boundary are employed in the computations. (b) Field distributions when the different radii of hypothetical boundary are employed.

defined by the subvectors X_1, X_2.[8,9] Because of a requirement of the specific shape of the hypothetical boundary, Eq. (7) can become a king-size system of equations. Thereby, the SDI solution becomes an expensive task. This deficiency can be removed by employing the VD transformation method.

In order to demonstrate the effectiveness of our method, we computed a simple magnetostatic field caused by the bifilar conductors. As shown in Fig. 3(a), much improved functionals were obtained by our VD transformation method when compared with those of the traditional first-order triangular finite-element method. This means that the size of the system of Eq. (7) may be considerably reduced to a small system so that the open fields can be evaluated in an efficient manner. Figure 3(b) shows the field distributions due to the bifilar conductors. The result in Fig. 3(b) reveals that the fairly good local solutions can be obtained from a small system. Figure 4(a) shows the convergence property of the functionals when different radii of hypothetical boundary are employed. Also, Fig. 4(b) shows that the fields are continuously distributed even if the different radii of hypothetic boundary are employed. The results of Fig. 4 suggest that a unique open-boundary solution can be obtained by the SDI method. Thus, the SDI solution requires a half of the overall computing cost needed by the simple dual energy approach.

III. CONCLUSION

As shown above, we have proposed a new efficient implementation technique of the SDI method based on the VD transformation. As a result, it has been shown that the SDI solution for the open-boundary problems can be obtained from a small size of the system equation without sacrificing the accuracy in functionals and local solutions.

[1] G. Meunier, J. L. Coulomb, S. J. Salon, and L. Krahenbul, IEEE Trans. Magn. **MAG-22**, 1040 (1986).

[2] C. A. Brebbia, Ed. *Progress in Boundary Element Methods* (Pentech, London, 1981).

[3] P. P. Silvester, D. A. Lowther, C. J. Carpenter, and E. A. Wyatt, Proc. IEE **124**, 1267 (1977).

[4] P. Bettess, Int. J. Num. Methods Eng. **11**, 53 (1977).

[5] H. Hurwitz, IEEE Trans. Magn. **MAG-20**, 1918 (1984).

[6] Y. Saito, K. Takahashi, and S. Hayano, IEEE Trans. Magn. **MAG-23**, 3569 (1987).

[7] Y. Saito, K. Takahashi, and S. Hayano, IEEE Trans. Magn. **MAG-24**, 2946 (1988).

[8] Y. Saito, K. Takahashi, and S. Hayano, *Proceedings of the First International Symposium on Applied Electromagnetics in Materials* (Pergamon, New York, 1989), p. 237.

[9] Y. Saito, Y. Nakazawa, and S. Hayano, Paper of the Technical Meeting on Magnetics, IEEJ, MAG-88-99 (unpublished).

[10] Y. Saito, S. Ikeguchi, and S. Hayano, *Proceedings of the First International Symposium on Applied Electromagnetics in Materials* (Pergamon, New York, 1989), p. 271.

[11] Y. Saito, S. Ikeguchi, and S. Hayano, Proc. PIERS (to appear).

[12] Y. Saito, Y. Kishino, S. Hayano, N. Tsuya, and Z. J. Cendes, IEEE Trans. Magn. **MAG-22**, 1057 (1986).

[13] Y. Saito, Y. Kishino, M. Fukushima, S. Hayano, and N. Tsuya, J. Appl. Phys. **63**, 3174 (1988).

[14] J. Pennman and J. R. Fraser, IEEE Trans. Magn. **MAG-18**, 319 (1982).

[15] F. F. Kuo, *Network Analysis and Synthesis* (Wiley, New York, 1966).