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Data representation by field calculus and leading to the orthonormal linear transforms

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Abstract. This paper proposes a method of data set representation by potential field equations. The principal idea of the representation is to regard a numerical data set as a kind of potential/source fields, leading to the governing equations. The modal analysis of the equations derives the orthonormal linear transform matrix. It is described that the matrix corresponds to that of discrete wavelets performing an efficient wavelet multi-resolution analysis.

1. Introduction

The spread of high performance and reasonably priced computers has yielded a large scale Internet community as well as information resources. Data handling technologies based on the digital computers are of main importance to realize more efficient networking and computing. Discrete wavelet transform (WT) becomes a deterministic methodology to handle the digital signals and images, e.g., compressing data quantity, extracting their characteristics, etc. [1]. Moreover, their applications to electromagnetic field calculation, solving for forward and inverse problems, have been investigated and spurred to faster calculation algorithm [2,3]. The conventional WT, however, sometimes suffers from limitation on subject data length which must be to the power of 2. Thereby, the applications depend on employed wavelet basis, and need an enormous memory installation for implementation. The aim of this study is to propose a new concept of representation to handle a numerical data set in most efficient mannar. As a result, the wavelet-like linear transform matrices are derived from the modal analysis of potential field equations. This paper proposes the modal-wavelet transform (MWT) as one of the WTs. The bases of MWT are derived from a modal analysis of the field of equations. Regarding a numerical data set as a potential field leads to a partial-differential-equation-based data modeling, i.e., the data set can be represented by Poisson's equations. Then, the modal analysis of the discretized Poisson's equation gives a modal matrix constituting characteristic vectors. The modal matrix enables us orthogonal transformation in the same way as WT. MWT uses this matrix as a wavelet basis. Because of the differential equation based modeling, MWT yields an optimal basis to the subject data length. We demonstrate two types of MWT based on differential equation and Green-function. One dimentional Fourier analysis to each of column vectors in the transformation matrix shows that MWT has the similar nature to Fourier transform not having complex numbers.

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Fig. 1. Source density representation of a 2D image and image recovery from the image source density (128×128 pixels). (a) Original image. (b) An example of source density. (c) Recovered image by Poisson equation. (d) Recovered image by Green function.

2. Data representation by field calculus

2.1. Data governing equations

We consider a discrete data modeling based on the classical field theory. Namely, a numerical data set is assumed to be the potential or source fields. According to the field theory a scalar field u caused by source density σ could be obtained by a solution of Poisson equation or a fundamental solution with green functions:

$$\varepsilon \nabla^2 u = -\sigma \quad \text{or} \quad u = \frac{1}{\varepsilon} \int g(r) \sigma dr, \tag{1}$$

where ε is the medium parameter of the field. Moreover, $g(r)$ and r denote a Green function and the distance from the source to reference points, respectively. Practical data set **U** is represented by following system of equations:

$$L\mathbf{U} = \mathbf{f} \quad \text{or} \quad G\mathbf{f} = \mathbf{U}, \tag{2}$$

where **f** and **U** denote the vectors corresponding to the source density σ and the scalar field u; L and G denote the coefficient matrices derived from the Laplacian operator and a Green function in Eq. (1), respectively. Our data representation employs the governing equations of potential fields like above. Since the calculus gives solutions as the functions, then it is possible to arbitrarily change the resolution of data sets, i.e., number of the data, from the solution of Eq. (2).

As an example, let each pixel value in Fig. 1(a) be a scalar potential assuming the medium parameter ε to be a constant on the entire field, then applying L or G^{-1} to Fig. 1(a) yields the source density distribution like Fig. 1(b). Solving Eq. (2) with the source density as vector **f** reproduces the image as in Figs 1 (c) and (d).

Concretely, Figs 1(a) and (c) are identical in values. (About detail of the image generation, please see reference [4].) Therefore, our discrete data modeling based on field equation is capable of representing numerical data sets.

2.2. Modal-wavelet transforms

As is well known, the matrices L and G in Eq. (2) derived by available discretizing methods, e.g., finite elements, etc., become the symmetrical as well as positive definite matrices. In case when the vector **U** has q elements, it is possible to obtain the characteristic values λ_i, $i = 1, 2, \cdots, q$, of the matrices L and G

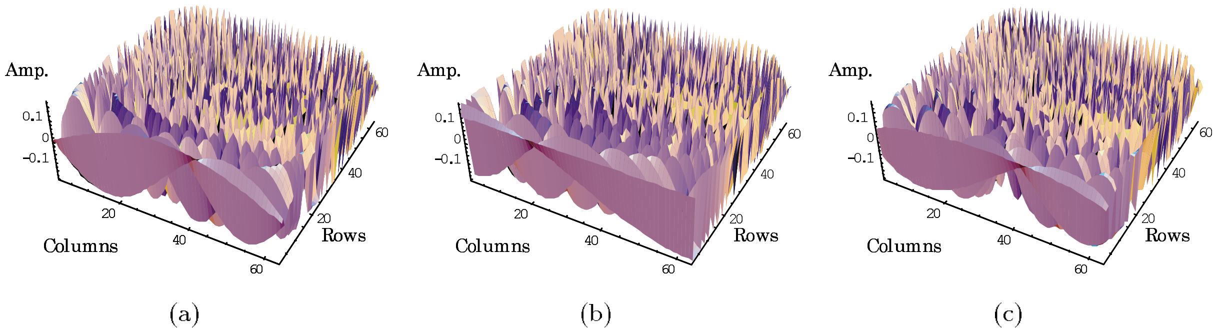

Fig. 2. Modal-wavelet matrices (64 × 64). (a) Dirichlet type boundary condition. (b) Neumann type boundary condition. (c) Green function type.

and their respective characteristic vectors $\mathbf{v}_i$, $i = 1, 2, \cdots, q$. The matrix composed of the characteristic vectors $\mathbf{v}_i$, $i = 1, 2, \cdots, q$ as its columns is called the modal matrix:

$$M_q = \left(\mathbf{v}_1\, \mathbf{v}_2 \cdots \mathbf{v}_q\right). \tag{3}$$

Because of the orthogonality, the following relationship holds

$$M_q^T M_q = I_q, \tag{4}$$

where the superscript T refers to a matrix transpose and I_q is a q by q identity matrix. The modal matrix derived from the coefficient matrix L or G has the same nature as those of the conventional WT matrices. Moreover, a linear combination of the characteristic vectors is possible to represent the value distribution in a data set. Thus, MWT employs this modal matrix as WT matrices.

2.3. *Modal-wavelet transform matrix and basis*

The MWT matrices can be derived various methods of discretizations. The MWT matrices introduced in the present paper are classified into two types. One is differential equation type assumed the subject data to be a potential field. The other is integral expression type assumed the subject data to be the field source distribution.

At first, let us consider MWT derived from differential equation. The simplest system matrix L can be obtained by one-dimensional Laplacian operation with equi-meshed three points finite difference approximation. Namely, the matrix L in Eq. (2) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} \simeq U_{x-1} - 2U_x + U_{x+1}, \quad x = 1, 2, \cdots, q. \tag{5}$$

Then, applying the Jacobi method yields a modal matrix M_q in Eq. (3). Therefore, the dimension of matrix M_q depends on number of subdivision of Eq. (5). This means it is possible to generate an optimal basis to the subject data. In the Laplace partial differential equation, two types of boundary conditions should be considered, i.e., the Dirichlet- and Neumann- type boundary conditions. Figures 2(a) and (b) illustrate the typical differential equation type MWT matrices. As shown in Figs 3 and 4, the bases having the Dirichlet- and Neumann- type boundary conditions become odd- and even- functions, respectively. The bases of MWT look like sinusoidal functions, however, the bases are composed of the multiple

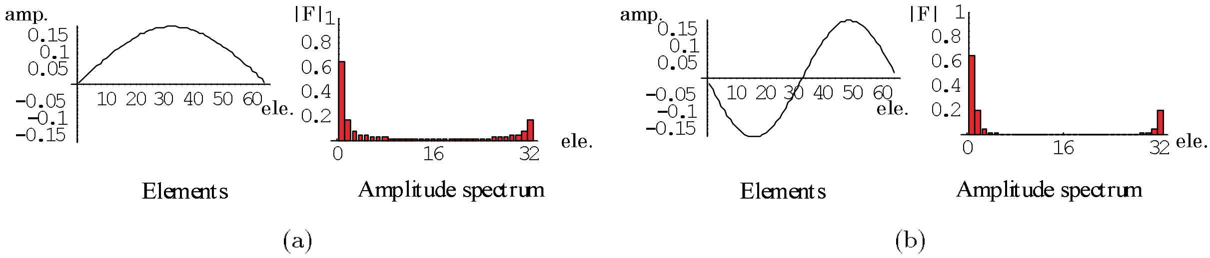

Fig. 3. Elements of row vectors in the matrix shown in Fig. 2(a) and their Fourier amplitude spectrum. (a) The first row vector. (b) The second row vector.

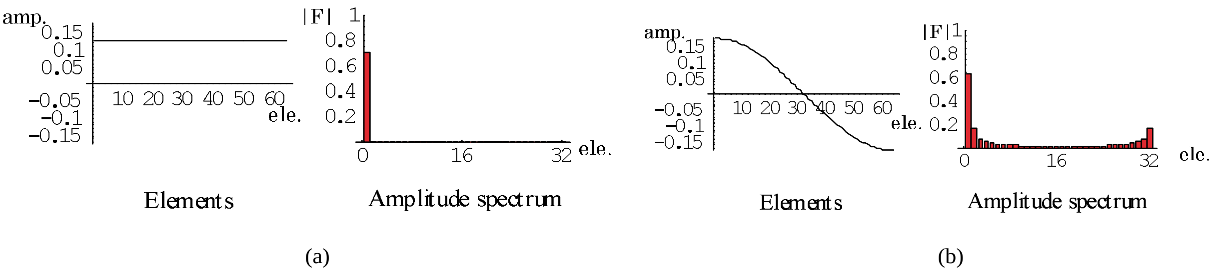

Fig. 4. Elements of row vectors in the matrix shown in Fig. 2(b) and their Fourier amplitude spectrum. (a) The first row vector. (b) The second row vector.

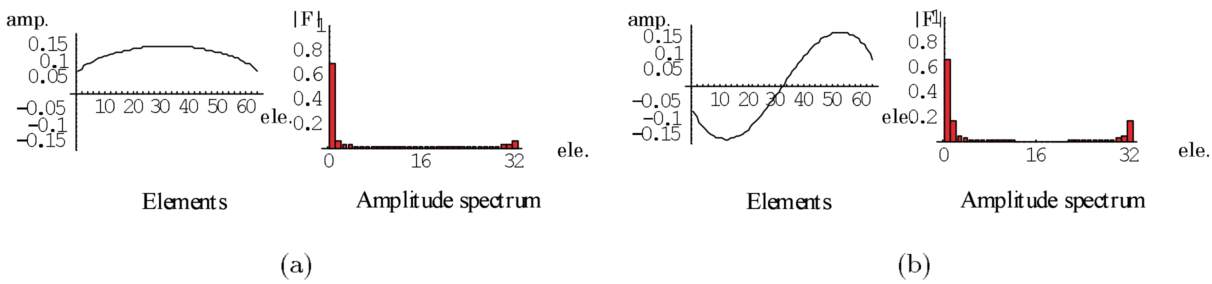

Fig. 5. Elements of row vectors in the matrix shown in Fig. 2(c) and their Fourier amplitude spectrum. (a) The first row vector. (b) The second row vector.

frequency components. Moreover, the elements constituting the transform matrices never become the complex numbers like in the Fourier transform.

Second, let us consider MWT derived from integral expression. We consider a three- dimensional Green function $g(r)$ in Eq. (1). However, the three-dimensional Green function takes infinity when $g(0)$ due to integral kernel. In order to remove this difficulty the matrix G in Eq. (2) is given by assuming the minimum distance $r_{i,i} = 1$, since the equi-grid image matrix is considered. Thus, we have

$$g(r) \simeq \begin{cases} 1/r_{i,j} & i \neq j \\ & \text{if} \\ 1 & i = j \end{cases}, \quad i = 1, 2, \cdots, q, \quad j = 1, 2, \cdots, q. \tag{6}$$

where the subscripts i and j refer to the source and reference points, respectively. Thereby, $r_{i,j}$ represents the distance between them. Since the system matrix derived from Eq. (6) becomes symmetrical, then the Jacobi method can be applied to obtain its modal matrix in much the same way as the MWT based on differential equation. Figures 2(c) and 5 show the MWT matrix and its bases. They have the similar patterns to that of the MWT matrix derived under the Dirichlet boundary condition.

3. Conclusions

We have proposed MWT derived from the data representation by equations for potential field. As shown above, MWT gives the transformation matrix having arbitrary dimension, so that it is possible to provide an efficient wavelet analysis from a viewpoint of memory consumption. Our approach has versatile capability not only to information resource handling but also smart computing.

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